Engineering Notes

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The Symmetric Far Wake Behind a Cylinder Beneath the Free Surface

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Nomenclature

a = circular cylinder radius
A_n = coefficients in series expansion for complex potential
B = multiplicative constant in asymptotic complex velocity expression
c = chord length
C_D = drag coefficient
d = constant in asymptotic complex velocity expression
E_1 = exponential integral
f = complex potential
f_B = complex perturbation potential in absence of free surface
f_F = complex potential satisfying free surface condition
F = Froude number
g = gravitational acceleration
h = height of undisturbed free surface
i = $(-1)^{1/2}$
I = exponential function $I(r) = e^{-r}E_1(-r)$
K = constant in Prandtl's mixing length hypothesis
l = characteristic wake width
p = function in similar solution for velocity defect
r,s = dummy variables
R = Reynolds number
t = body thickness
u,v = velocity components in x and y directions
U = freestream speed
U_e = wake centerline velocity
w = maximum velocity defect
W = complex velocity
x,y = coordinates along and normal to chord
$\bar{x}$ = x/x_0
z = complex variable $z = x + iy$
δ^* = displacement thickness
ϵ = thickness parameter $\epsilon = t/h$
μ = viscosity coefficient
ν = wave number $\nu = g/U^2$
ρ = density
η = similarity variable $\eta = y/l(x)$

Subscript

0 = initial conditions for viscous solution

THERE is much interest in the interaction of the effects of viscosity and free-surface waves in the determination of the resistance to the motion of a body in or beneath the free surface. Recently, Gebhardt[1] has studied the effect of the free surface on the development of the viscous boundary layer on a submerged two-dimensional hydrofoil. Only the boundary layer on the body is considered. This note will present results for the symmetric wake far downstream of a cylinder beneath the free surface and immersed in an incompressible stream.

Inviscid Pressure Distribution

Consider the flow of a stream of speed U in the x-direction past a symmetric two-dimensional body submerged a distance h beneath the undisturbed free surface. The coordinate system lies in the undisturbed free surface with y positive up. The characteristic thickness of the body is t and the characteristic length c is $0(t)$.

Salvesen[2] shows that, to first order in $\epsilon = t/h$, the complex potential in the far field is given by

$$f = Uz + f_B + f_F$$

where $z = x + iy$, f_B is the complex perturbation potential in the absence of the free surface, and f_F satisfies the first-order linearized free-surface condition.

In the absence of the free surface, far from the body the complex potential can be written as

$$f_B = \sum_2^\infty \frac{A_n}{(z+ih)^{n-1}} = \frac{A_2}{z+ih} + \frac{A_3}{(z+ih)^2} + \ldots$$

where the A_2 term represents a doublet and to first order the circulation is zero.

To obtain f_F, let us first consider the doublet term

$$f_B = \frac{A_2}{z+ih}$$

For this case, we see from Wehausen and Laitone[3] that

$$f_F = -A_2\left\{2i\,\nu I[(i\nu(z-ih)] + \frac{1}{z-ih}\right\}$$

where the nondimensional wave number $t\nu = tg/U^2$ is $0(1)$ and the exponential function

$$I(r) = e^{-r}E_1(-r) = e^{-r}\int_{-r}^{\infty} e^{-s}ds/s$$

where E_1 is the exponential integral. We can obtain $f_B + f_F$ for the higher-order singularities by successively differentiating the doublet results and making use of

$$\frac{d}{dz}I[i\nu(z-ih)] = -i\nu I[i\nu(z-ih)] - \frac{1}{z-ih}$$

Far downstream, the following asymptotic expansion is valid

$$I(r) \sim \left(\frac{1}{r} + \frac{1}{r^2} + \frac{2!}{r^3} + \ldots\right) + 2i\pi e^{-r}$$

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Therefore far downstream, the only terms which will remain in the perturbation potential are those involving I, and specifically only the last term in I will remain. The complex potential far behind the body is

$$f = Uz + 4\pi\nu e^{-i\nu(z-ih)}\left[A_2 + \sum_3^\infty \frac{A_n}{n-2}(-i\nu)^{n-2}\right]$$

The complex velocity $u - iv$ is given by

$$W = \frac{df}{dz} = U - 4\pi i\nu^2 e^{-i\nu(z-ih)}\left[A_2 + \sum_3^\infty \frac{A_n}{n-2}(-i\nu)^{n-2}\right]$$

For convenience, let

$$-\frac{4\pi i\nu^2}{U}\left[A_2 + \sum_3^\infty \frac{A_n}{n-2}(-i\nu)^{n-2}\right] = Be^{i\nu d}$$

Then

$$W/U = 1 + Be^{-i\nu(z-ih-d)}$$

Along the wake centerline, $y = -h$, the tangential velocity component $u = U_e(x)$ is

$$\frac{U_e}{U} = 1 + Be^{-2\nu h}\cos\nu(x-d)$$

We note that $\nu h = 0(1/\epsilon)$ and that the result differs from the infinite fluid case, $U_e = U$, by a small wave-like component. This is the same result as that for the flow past a wavy wall with wave number ν and amplitude B/ν (see Liepmann and Roshko[4]). This result is not unexpected since the flow behind a two-dimensional submerged body may be represented by a Stokes wave.

As an example, let us consider the flow past a circular cylinder of radius a. In this case, $A_2 = Ua^2$ and $A_n = 0$ for $n > 2$. Then

$$B = 4\pi a^2\nu^2 = 4\pi F^{-4}$$

where $F = U/(ga)^{1/2}$ is the Froude number based on cylinder radius. It has already been assumed that $F = 0(1)$. The maximum deviation of the wake centerline velocity from its infinite medium value is

$$Be^{-2\nu h} = 4\pi F^{-4}e^{-2(h/a)F^{-2}}$$

Viscous Wake

We will use the analysis of Prabhu[5] who has solved for the incompressible far wake with arbitrary pressure gradient. It is convenient for the viscous analysis to move our origin of coordinates to the more conventional location at the body. The boundary-layer equations for two-dimensional incompressible flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_e\frac{dU_e}{dx} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

where μ/ρ is the kinematic viscosity in laminar flow and the eddy viscosity (assumed a function of x only) in turbulent flow. The wake boundary conditions are

$$v = \partial u/\partial y = 0 \text{ at } y = 0, \quad u \to U_e \text{ as } y \to \infty$$

A similar solution of the form

$$u(x,y) = U_e(x) + w(x)p(\eta)$$

is assumed where $w(x)$ is the maximum velocity defect, $\eta = y/\ell(x)$ and $\ell(x)$ is a characteristic wake width, and $w/U_e \ll 1$. Then

$$p = -e^{-\eta^2 \log 2}$$

where ℓ is now defined as the distance between the wake centerline and the point where the velocity defect is half of its maximum value.

Laminar Flow

We introduce the arbitrary initial conditions $U_e = U_0$, $\ell = \ell_0$, $w = w_0$ at $x = x_0$. The solutions for ℓ and w are then

$$\frac{\ell}{\ell_0} = \frac{U_0}{U_e}\left[1 + \frac{4\log 2}{R_0}\int_{\tilde{x}_0}^{\tilde{x}} \frac{U_e}{U_0}d\tilde{x}\right]^{1/2}$$

$$\frac{w}{w_0} = \frac{U_0}{U_e}\left[1 + \frac{4\log 2}{R_0}\int_{\tilde{x}_0}^{\tilde{x}} \frac{U_e}{U_0}d\tilde{x}\right]^{-1/2}$$

where $\tilde{x} = x/\ell_0$ and $R_0 = \rho U_0\ell_0/\mu$.

In the absence of the free surface, $U_e = U_0 = U$ and for large x, the displacement thickness,

$$\delta^* = \int_0^\infty (1 - u/U_e)dy$$

is given by

$$\frac{\delta^*}{t} = \frac{w_0\ell_0}{tU}\left(\frac{\pi\log 2}{4}\right)^{1/2} = \frac{C_D}{4} = \text{constant}$$

where C_D is the drag coefficient. Also, for large x

$$\ell/t \sim R^{-1/2}(x/t)^{1/2}$$

and

$$w/U \sim C_D R^{1/2}(t/x)^{1/2}$$

where R is the body Reynolds number and C_D is $0(R^{-1/2})$.

With the free surface, we use the pressure distribution previously calculated and for convenience choose $U_0 = U$. Then

$$\frac{\ell}{\ell_0} = \left[1 + \frac{4\log 2}{R_0}\left\{\frac{x-x_0}{\ell_0} + \frac{Be^{-2\nu h}}{\nu\ell_0}[\sin\nu(x-d) - \sin\nu(x_0-d)]\right\}\right]^{1/2} / [1 + Be^{-2\nu h}\cos\nu(x-d)]$$

and

$$\frac{w}{w_0} = \left[1 + \frac{4\log 2}{R_0}\left\{\frac{x-x_0}{\ell_0} + \frac{Be^{-2\nu h}}{\nu\ell_0}[\sin\nu(x-d) - \sin\nu(x_0-d]\right\}\right]^{-1/2} / [1 + Be^{-2\nu h}\cos\nu(x-d)]$$

For large x,

$$\frac{\delta^*}{t} = (C_D/4)/[1 + Be^{-2\nu h}\cos\nu(x-d)]$$

$$\sim (C_D/4)[1 - Be^{-2\nu h}\cos\nu(x-d)]$$

$$\frac{w}{U} \sim C_D R^{1/2}(t/x)^{1/2}[1 - Be^{-2\nu h}\cos\nu(x-d)]$$

$$\frac{\ell}{t} \sim R^{-1/2}(x/t)^{1/2}[1 - Be^{-2\nu h}\cos\nu(x-d)]$$

The free surface results for displacement thickness, velocity defect and wake width oscillate about the infinite medium results. The magnitude of the oscillations is dependent on F and ϵ.

Turbulent Flow

To arrive at an approximate solution for the turbulent flow case, we make use of Prandtl's mixing length hypothesis and assume as in Prabhu that $\mu/\rho = Kw\ell K$ is constant. Then

$$\frac{\ell}{\ell_0} = \frac{U_0}{U_e}\left[1 + 4K\frac{w_0}{U_0}\log 2\int_{\tilde{x}_0}^{\tilde{x}}\frac{U_0}{U_e}d\tilde{x}\right]^{1/2}$$

$$\frac{w}{w_0} = \frac{U_0}{U_e}\left[1 + 4K\frac{w_0}{U_0}\log 2\int_{\tilde{x}_0}^{\tilde{x}}\frac{U_0}{U_e}d\tilde{x}\right]^{-1/2}$$

We therefore see that the results will have the same character as those for the laminar case. The free-surface results will oscillate about the infinite medium results.

References

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